Analytical Stability Condition of the Latency Insertion Method for Nonuniform G*LC* Circuits

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***Abstract*—The latency insertion method (LIM) is a transient simulation technique for circuits and is based on a finite-difference formulation, like the well-known finite-difference time-domain (FDTD) method for solving Maxwells equations. The LIM, like the FDTD method, is only conditionally stable resulting in an upper bound for the time step of the transient simulation. This bound on the time step is a function of the circuit topology and the circuit element values. It is critical to know this bound analytically for a given circuit. However, stability conditions of the LIM have been proven only for 1-D, infinitelylong, distributed uniform *RLC* circuits, employed in transmission line modeling. For nonuniform circuits, these conditions have been predicted and have been observed experimentally as well but have not been possible to prove using the existing stability analysis techniques. Recently, analytical stability conditions of the LIM for nonuniform *RLC* circuits have been proven using the Lyapunovs direct method (LDM). However, when a conductance to ground (G) is added to a node of an *LC* or *RLC* circuit, the stability conditions cannot be derived using the Lyapunov function proposed. In this brief, analytical stability condition of the LIM is derived for the first time for nonuniform G*LC* circuits using the LDM with a new Lyapunov function.**

***Index Terms*—Conditional stability, finite-difference time-domain (FDTD) method, spectral radius.**

I. Introduction

The rest of the paper is organized as follows. In Section II, the latency insertion method (LIM) formulation for the G*LC* circuits is described. In Section III, the origin of the conditional stability in the LIM is described. Also, in this section, the problem solved in this paper is defined mathematically. In Section IV, the Lyapunov's direct method (LDM) is described. In Section V, analytical stability conditions of LIM for nonuniform G*LC* circuits are derived using LDM. In Section VII, the analytical stability condition is numerically verified. Finally, in Section VIII, the conclusions of this work are drawn.

II. LIM-Based Transient Simulation Formulation for Nonuniform G*LC* Circuits

An example of a nonuniform G*LC* circuit is shown in Fig. 1. Some transmission lines can be modeled using a distributed G*LC* circuit derived from Fig. 1. Each inductor in this circuit is defined as a branch. Each node is marked as a solid black circle. The suffixes i and b denote a node and a branch, respectively. The quantity L_b denotes the inductance of branch b; the

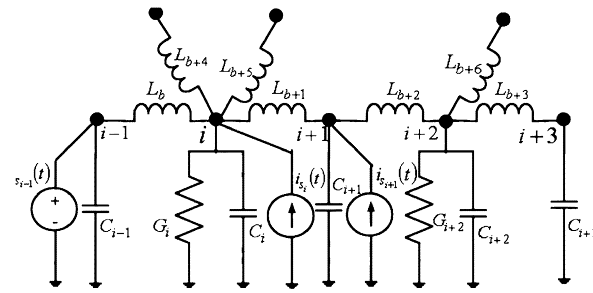

Fig. 1. Example of a nonuniform G*LC* circuit.

quantities C_i and G_i denote the capacitance to ground and conductance to ground from node i, respectively. To enable LIM, $L_b > 0$ and $C_i > 0$ (see [1]). The quantity $G_i \geq 0$. Let N_b^i denote the number of branches connected to node i. In a uniform G*LC* circuit, there are some restrictions on the circuit elements's values and on the circuit topology: All branches should have the same inductance, i.e., $L_b = L$ for all b's, and all nodes should have the same capacitance to ground and the same conductance to ground, i.e., $C_i = C$ and $G_i = G$ for all i's. Moreover, each node is connected to same number of branches, i.e., $N_b^i = N$ for all i. In a nonuniform G*LC* circuit, there are no such restrictions (see Fig. 1). Moreover, the quantity N_b^i can be any positive integer. In Fig. 1, the quantity $i_{s_i}(t)$ denotes a transient current source connected to node i, and $v_{s_i}(t)$ a transient voltage source connected to node i. The objective is to compute the transient node voltages computationally efficiently.

LIM is a transient simulation algorithm for circuits, similar to the finite-difference time-domain (FDTD) method [2] for dielectric media, and has optimal computational efficiency [1]. The LIM formulation for the transient simulation in G*LC* circuits (see Fig. 1) is described next. Let N_n denote the number of nodes and N_b the number of branches. Let $\mathbf{C} \in \mathbb{R}^{N_n \times N_n}$ and $\mathbf{G} \in \mathbb{R}^{N_n \times N_n}$ denote the diagonal matrices of C_i's and G_i's, respectively. Let $\mathbf{L} \in \mathbb{R}^{N_b \times N_b}$ be the corresponding diagonal matrix of branch inductances. Let $v_i^{n+(1/2)}$ be the voltage of node i at time instant $(n+(1/2))\Delta t$, and let $\mathbf{v}^{n+(1/2)} \in \mathbb{R}^{N_n \times 1}$ be the vector of node voltages. Similarly, i_b^n be the current in branch b at time instant $n\Delta t$, and let $\mathbf{i}^n \in \mathbb{R}^{N_b \times 1}$ be the vector of branch currents. Let $\mathbf{i_s}(t) \in \mathbb{R}^{N_n \times 1}$ be the vector of source currents entering nodes at time t. The LIM formulation involves obtaining update expressions for node voltages from the Kirchoff's current laws (KCL) (at nodes) and update expressions for branch currents from the Kirchoff's voltage laws (KVL) (in branches) in a Yee-FDTD [2] manner.

The KCL at all nodes can be written as

$$\mathbf{C}\dot{\mathbf{v}}(t) + \mathbf{G}\mathbf{v}(t) = -\mathbf{M}^T\mathbf{i}(t) + \mathbf{i_s}(t) \quad (1)$$

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where $\overset{\bullet}{v}(t) = dv(t)/dt$, the quantity $\mathbf{M}^T$ is the transpose of $\mathbf{M}$, and $\mathbf{M} \in \mathbb{Z}^{N_b \times N_n}$ is the edge-to-node incidence matrix. An entry in $\mathbf{M}$ corresponding to branch b and node i is defined as

$$\mathbf{M}(b,i) = \begin{cases} 1, & \text{if } i_b \text{ is flowing out of node } i \\ -1, & \text{if } i_b \text{ is flowing into node } i \\ 0, & \text{otherwise.} \end{cases}$$

When the KCLs in (1) are discretized using a semi-implicit integration scheme [3], the equation

$$\mathbf{C}\frac{\mathbf{v}^{n+\frac{1}{2}} - \mathbf{v}^{n-\frac{1}{2}}}{\Delta t} + \mathbf{G}\frac{\mathbf{v}^{n+\frac{1}{2}} + \mathbf{v}^{n-\frac{1}{2}}}{2} = -\mathbf{M}^T\mathbf{i}^n + \mathbf{i}_\mathbf{s}^n \quad (2)$$

can be obtained. From (2), the node voltages are updated using the expression

$$(\mathbf{C}+0.5\Delta t\mathbf{G})\mathbf{v}^{n+\frac{1}{2}} = (\mathbf{C}-0.5\Delta t\mathbf{G})\mathbf{v}^{n-\frac{1}{2}} - \Delta t\mathbf{M}^T\mathbf{i}^n + \Delta t\mathbf{i}_\mathbf{s}^n. \quad (3)$$

Following a similar procedure, an update expression for the branch currents can be obtained. The KVLs in branches can be written as

$$\mathbf{L}\,\overset{\bullet}{\mathbf{i}}\,(t) = \mathbf{M}\mathbf{v}(t) \quad (4)$$

the discretized version of (4) can be written as

$$\mathbf{L}\frac{\mathbf{i}^{n+1} - \mathbf{i}^n}{\Delta t} = \mathbf{M}\mathbf{v}^{n+\frac{1}{2}} \quad (5)$$

and the update expression for the branch currents can be obtained from (5) as

$$\mathbf{L}\mathbf{i}^{n+1} = \mathbf{L}\mathbf{i}^n + \Delta t\mathbf{M}\mathbf{v}^{n+\frac{1}{2}}. \quad (6)$$

The transient simulation using LIM involves computing the node voltages using (3) first and computing the branch currents using (6) next for each time step. When a node is connected to a voltage source, then as an intermediate step, the voltage of this node is made equal to the value of the voltage source at the current time instant.

The LIM transient simulation has optimal memory and time complexity (note the update process (3) and (6) involves only diagonal matrices) and has $O((\Delta t)^2)$ accuracy (see [1]) However, this simulation is stable only for restricted values of Δt, i.e., the LIM formulation is only conditionally stable [1], [3], [4].

III. Conditional Stability and Stability Analysis of LIM

The discrete system described by (3) and (6) can be rewritten as

$$\mathbf{u}^{n+1} = \mathbf{A}^{-1}\mathbf{B}\mathbf{u}^n + \mathbf{A}^{-1}\mathbf{r}^n \quad (7)$$

where the term $\mathbf{u}^n$ is the state vector, defined as

$$\mathbf{u}^n = \left[(\mathbf{i}^n)^T \quad \left(\mathbf{v}^{n-\frac{1}{2}}\right)^T\right]^T$$

the term $\mathbf{r}^n$ is the input vector, defined as

$$\mathbf{r}^n = \left[(\mathbf{0})^T \quad \Delta t\,(\mathbf{i}_\mathbf{s}^n)^T\right]^T, \quad \mathbf{A} = \begin{bmatrix} \mathbf{L} & -\Delta t\mathbf{M} \\ \mathbf{0} & \mathbf{C}+0.5\Delta t\mathbf{G} \end{bmatrix} (8)$$

$$\mathbf{B} = \begin{bmatrix} \mathbf{L} & \mathbf{0} \\ -\mathbf{\Delta t M}^T & \mathbf{C}-0.5\Delta t\mathbf{G} \end{bmatrix}. \quad (9)$$

The stability of the discrete system can be defined [5]: 1) based on the boundedness of state, $\mathbf{u}^n$, given an initial condition for the state and a zero input, $\mathbf{r}^n$; 2) based on the bounded input bounded state (BIBS) stability; and 3) based on the bounded input bounded output (BIBO) stability. The focus of this paper is on the first kind of stability. This kind of stability is also a necessary condition for the BIBS stability [5], as it is a special case of the latter with $\mathbf{r}^n = \mathbf{0}$.

For the state stability, all the eigenvalues of the matrix $\mathbf{A}^{-1}\mathbf{B}$ should have a magnitude less than or equal to unity. In other words, the spectral radius of $\mathbf{A}^{-1}\mathbf{B}$, $\rho(A^{-1}B)$, is less than or equal to one. Unfortunately, the eigenvalues of $\mathbf{A}^{-1}\mathbf{B}$ depend on Δt, resulting in a conditional stability of (7) dictated by the choice of Δt.

The conditions on Δt can be computed by requiring $\rho(\mathbf{A}^{-1}\mathbf{B}) \leq 1$. However, finding eigenvalues of $\mathbf{A}^{-1}\mathbf{B}$ analytically is a difficult problem. This difficulty is avoided in some circuit toplogies if von Neumann method [6] is used for stability analysis [3]. In this method, the conditions on Δt are determined by requiring the fourier amplitude of the state vector to be bounded by unity. The need to analyze the system in the Fourier domain requires the circuit element values to be equal and the circuit topology to be uniform at every point in the circuit; moreover, the circuits have to infinitely long. Specifically, analytical condition on Δt is known and proven only for 1-D (i.e., $N_b^i = 2$) uniform *RLC* circuit (similar to uniform G*LC* circuit). Therefore, for a nonuniform G*LC* circuit, the stability condition cannot be derived using VM.

In [7], a similar problem in the FDTD method is solved for nonuniform lossy dielectric media. The approach in [7] is based on LDM, introduced to the FDTD community in [8].

There are three important differences between the LIM problem and the FDTD problem [7] and [8]: 1) LIM discretizes only the circuits even when the circuits are discontinuous, while the FDTD problem discretizes both the dielectric medium and the free space. Therefore, the FDTD problem always solves a continuous problem domain. 2) Unlike the FDTD problem, the LIM problem can have more than three dimensions: in the LIM problem, the dimensions weakly refer to the number of branches connected to a node, which can be more than three. 3) Unlike the FDTD problem, the circuit problem is nonuniform with respect to the number of branches connected to a node.

In [4], the approach in [7] is employed to derive the stability condition of LIM for nonuniform *RLC* circuit. However, the Lyapunov function employed in [4] does not remain a Lyapunov for G*LC* circuits. The objective of this paper is to obtain the conditions on Δt for the state stability of LIM for a nonuniform G*LC* circuit using the Lyapunov's direct method, discussed next for a discrete-time system.

IV. LDM for Discrete-Time System

The stability of discrete-time systems can be analyzed using Lyapunov's direct method [5], which can be stated as follows:

Let $\mathbf{u} \in \mathbb{R}^n$ be a vector of states of system, and $\mathbf{u} = \mathbf{0}$ be the equilibrium point. Suppose there exists a scalar function $E(\mathbf{u})$ continuous in $\mathbf{u}$ such that

$$E(\mathbf{0}) = 0, \quad E(\mathbf{u}) > 0, \qquad \text{for} \quad \mathbf{u} \neq \mathbf{0} \tag{10}$$
$$E(\mathbf{u}(n\Delta t)) - E(\mathbf{u}(n-1)\Delta t) \leq 0, \qquad \text{for all } \mathbf{u}. \tag{11}$$

Then, $\mathbf{u} = \mathbf{0}$ is stable. Moreover, if

$$E(\mathbf{u}(n\Delta t)) - E(\mathbf{u}(n-1)\Delta t) < 0 \quad \text{for } \mathbf{u} \neq \mathbf{0} \tag{12}$$

then $\mathbf{u} = \mathbf{0}$ is asymptotically stable. If $E(\mathbf{u})$ satisfies (10) and (12) along with the condition that

$$||\mathbf{u}|| \longrightarrow \infty \Longrightarrow E(\mathbf{u}) \longrightarrow \infty \tag{13}$$

then $\mathbf{u} = \mathbf{0}$ is globally asymptotically stable. The symbol $||\mathbf{u}||$ in (13) stands for the p-norm of the vector $\mathbf{u}$, where $p = 1$, 2, and ∞. A continuous scalar function $E(\mathbf{u})$ satisfying (10) and (11) is called a Lyapunov function. Existence of a Lyapunov function is a sufficient condition for the stability of $\mathbf{u} = \mathbf{0}$.

V. Analytical Stability Condition for Nonuniform G*LC* Circuits

In this section, analytical stability condition for the LIM formulation in Section II is derived using LDM.

Since only the state stability is demonstrated, input (or excitations) are set to zero in (2). When $\mathbf{i_s}(t) = 0$, Equation (2) can be rewritten as

$$\mathbf{C}\frac{\mathbf{v}^{n+\frac{1}{2}} - \mathbf{v}^{n-\frac{1}{2}}}{\Delta t} = -\mathbf{M}^T\mathbf{i}^n - \mathbf{G}\frac{\mathbf{v}^{n+\frac{1}{2}} + \mathbf{v}^{n-\frac{1}{2}}}{2}. \tag{14}$$

For convenience, the discretized KVL (5) is repeated here

$$\mathbf{L}\frac{\mathbf{i}^{n+1} - \mathbf{i}^n}{\Delta t} = \mathbf{M}\mathbf{v}^{n+\frac{1}{2}}. \tag{15}$$

The equilibrium state of (14) and (15) is same as that of (7). This state for (7) is the state for which $\mathbf{u}^{n+1} = \mathbf{u}^n$ for all n in the absence of $\mathbf{r}^n$ (see [5], pp. 343). The origin $\mathbf{u_e} = \mathbf{0}$ is an equilibrium state of (7). In the following, an energy-like function is chosen as a scalar function, and the conditions for this function to be a Lyapunov function for (14) and (15) are determined. These conditions in turn result in an upper bound for Δt. When Δt is chosen within this upper bound, $\mathbf{u_e} = 0$ is stable.

First, the Lyapunov function proposed in [4] for a nonuniform *RLC* circuit is tested for its Lyapunov property for (14) and (15). This function is

$$F^n = \frac{1}{2}(\mathbf{i}^n)^T\mathbf{L}\mathbf{i}^n + \frac{1}{2}\left(\mathbf{v}^{n-\frac{1}{2}}\right)^T\mathbf{C}\mathbf{v}^{n+\frac{1}{2}}. \tag{16}$$

The function F^n in (16) is checked if it satisfies (11), a property of Lyapunov functions, for (14) and (15). Using (16), the difference $F^n - F^{n-1}$ can be written as

$$\begin{aligned}F^n - F^{n-1} =& \frac{1}{2}\Big[(\mathbf{i}^n)^T\mathbf{L}\mathbf{i}^n - (\mathbf{i}^{n-1})^T\mathbf{L}\mathbf{i}^{n-1} \\ &+ \left(\mathbf{v}^{n-\frac{1}{2}}\right)^T\mathbf{C}\mathbf{v}^{n+\frac{1}{2}} - \left(\mathbf{v}^{n-\frac{3}{2}}\right)^T\mathbf{C}\mathbf{v}^{n-\frac{1}{2}}\Big] \\ =& \frac{1}{2}\Big[(\mathbf{i}^n + \mathbf{i}^{n-1})^T\mathbf{L}(\mathbf{i}^n - \mathbf{i}^{n-1}) \\ &+ \left(\mathbf{v}^{n-\frac{1}{2}}\right)^T\mathbf{C}\left(\mathbf{v}^{n+\frac{1}{2}} - \mathbf{v}^{n-\frac{3}{2}}\right)\Big]\end{aligned}$$

which can be simplified using (14) and (15) as

$$F^n - F^{n-1} = -\frac{\Delta t}{4}\left(\mathbf{v}^{n-\frac{1}{2}}\right)^T\mathbf{G}\left(\mathbf{v}^{n+\frac{1}{2}} + 2\mathbf{v}^{n-\frac{1}{2}} + \mathbf{v}^{n-\frac{3}{2}}\right). \tag{17}$$

From the RHS of (17), the condition $F^n - F^{n-1} \leq 0$ can not guaranteed for all $\mathbf{v}$'s; therefore F^n does not satisfy (11). Consequently, the function F^n can not be a Lyapunov function for (14) and (15).

Since the existence of a Lyapunov function is only a sufficient condition for stability, the nonLyapunov nature of F^n for (14) and (15) does not imply the instability of $\mathbf{u_e} = \mathbf{0}$. Therefore, a new function E^n is chosen as a potential candidate for the Lyapunov function for (14) and (15):

$$E^n = \frac{1}{2}(\mathbf{i}^{n-1})^T\mathbf{L}\mathbf{i}^n + \frac{1}{2}\left(\mathbf{v}^{n-\frac{1}{2}}\right)^T\mathbf{C}\mathbf{v}^{n-\frac{1}{2}}. \tag{18}$$

The function E^n can be shown to satisfy the condition in (11): Using (18), the quantity $E^n - E^{n-1}$ can be written as

$$\begin{aligned}E^n - E^{n-1} =& \frac{1}{2}\Big[(\mathbf{i}^{n-1})^T\mathbf{L}\mathbf{i}^n - (\mathbf{i}^{n-2})^T\mathbf{L}\mathbf{i}^{n-1} \\ &+ \left(\mathbf{v}^{n-\frac{1}{2}}\right)^T\mathbf{C}\mathbf{v}^{n-\frac{1}{2}} - \left(\mathbf{v}^{n-\frac{3}{2}}\right)^T\mathbf{C}\mathbf{v}^{n-\frac{3}{2}}\Big] \\ =& \frac{1}{2}\Big[(\mathbf{i}^n + \mathbf{i}^{n-1})^T\mathbf{L}(\mathbf{i}^n - \mathbf{i}^{n-2}) \\ &+ \left(\mathbf{v}^{n-\frac{1}{2}} + \mathbf{v}^{n-\frac{3}{2}}\right)^T\mathbf{C}\left(\mathbf{v}^{n-\frac{1}{2}} - \mathbf{v}^{n-\frac{3}{2}}\right)\Big]\end{aligned}$$

which can be simplified using (14), (15) as

$$\begin{aligned}E^n - E^{n-1} =& -\frac{\Delta t}{4}\left(\mathbf{v}^{n-\frac{1}{2}} + \mathbf{v}^{n-\frac{3}{2}}\right)^T\mathbf{G}\left(\mathbf{v}^{n-\frac{1}{2}} + \mathbf{v}^{n-\frac{3}{2}}\right) \\ \leq& 0.\end{aligned} \tag{19}$$

The inequality in (19) is true as $\mathbf{G}$ is positive semidefinite (semidefinite because conductances can be zero). Additionally, if $\mathbf{G}$ is positive definite, E^n satisfies (12).

For E^n to satisfy (10), E^n is written as

$$\begin{aligned}E^n =& \frac{1}{2}(\mathbf{i}^{n-1})^T\mathbf{L}\mathbf{i}^{n-1} + \frac{1}{2}\left(\mathbf{v}^{n-\frac{1}{2}}\right)^T\mathbf{C}\mathbf{v}^{n-\frac{1}{2}} \\ &+ \frac{\Delta t}{2}(\mathbf{i}^{n-1})^T\mathbf{M}\mathbf{v}^{n-\frac{1}{2}} \\ =& \frac{1}{2}\begin{bmatrix}\mathbf{i}^{n-1} \\ \mathbf{v}^{n-\frac{1}{2}}\end{bmatrix}^T\begin{bmatrix}\mathbf{L} & \frac{\Delta t}{2}\mathbf{M} \\ \frac{\Delta t}{2}\mathbf{M} & \mathbf{C}\end{bmatrix}\begin{bmatrix}\mathbf{i}^{n-1} \\ \mathbf{v}^{n-\frac{1}{2}}\end{bmatrix} \\ =& \frac{1}{2}\mathbf{x}^T\mathbf{P}\mathbf{x}.\end{aligned} \tag{20}$$

For E_n to satisfy (10), the matrix $\mathbf{P}$ in (20) has to be positive definite. In the following, the conditions for $\mathbf{P}$ to be positive definite are found. The stability conditions are derived as a result. The stability conditions are found first when each node is connected to only two branches. These conditions are extended when the number of these branches is arbitrary and different for different nodes in the circuit.

A. Condition on Δt When Two Branches are Connected to Every Node

Let the subscript i denote a node, and let the subscripts $i-(1/2)$ and $i+(1/2)$ denote the two branches connected to node i. Let $i^n_{i-(1/2)}$ and $i^n_{i+(1/2)}$ denote the branch currents that enter and leave node i, respectively. Let b denote a branch, and let the two nodes of this branch be denoted by $(b,1)$ and $(b,2)$, with the branch current i^n_b flowing from node $(b,1)$ to node $(b,2)$. The quantity E^n in (20) can also be written as

$$\begin{aligned} E^n &= \frac{1}{2}\left[\sum_{b=1}^{N_b} L_b\left(i_b^{n-1}\right)^2 + \sum_{i=1}^{N_n} C_i\left(v_i^{n-\frac{1}{2}}\right)^2 \right.\\ &\qquad \left. +\Delta t\sum_{b=1}^{N_b} i_b^{n-1}\left(v_{b,1}^{n-\frac{1}{2}} - v_{b,2}^{n-\frac{1}{2}}\right)\right]\\ &= \sum_{b=1}^{N_b}\frac{1}{2}\left[L_b\left(i_b^n\right)^2 + \frac{C_{b,1}}{2}\left(v_{b,1}^{n-\frac{1}{2}}\right)^2 + \frac{C_{b,2}}{2}\left(v_{b,2}^{n-\frac{1}{2}}\right)^2\right.\\ &\qquad \left. +\Delta t i_b^n\left(v_{b,1}^{n-\frac{1}{2}} - v_{b,2}^{n-\frac{1}{2}}\right)\right]\\ &= \sum_{b=1}^{N_b} E_b^n|_{(2,2)} \end{aligned} \tag{21}$$

where

$$E_b^n|_{(p,q)} = \frac{1}{2}\left[L_b\left(i_b^n\right)^2 + \frac{C_{b,1}}{p}\left(v_{b,1}^{n-\frac{1}{2}}\right)^2 + \frac{C_{b,2}}{q}\left(v_{b,2}^{n-\frac{1}{2}}\right)^2 + \Delta t i_b^n\left(v_{b,1}^{n-\frac{1}{2}} - v_{b,2}^{n-\frac{1}{2}}\right)\right]. \tag{22}$$

From (21), E^n is positive (in other words satisfies (10)) if $E^n_b|_{(2,2)}$ is positive for all b. Expressing $E^n_b|_{(2,2)}$ as a quadratic form

$$\begin{aligned} E_b^n|_{(2,2)} &= \frac{1}{2}\begin{bmatrix} i_b^{n-1}\\ v_{b,1}^{n-\frac{1}{2}}\\ v_{b,2}^{n-\frac{1}{2}}\end{bmatrix}^T \begin{bmatrix} L_b & \frac{\Delta t}{2} & -\frac{\Delta t}{2}\\ \frac{\Delta t}{2} & \frac{C_{b,1}}{2} & 0\\ -\frac{\Delta t}{2} & 0 & \frac{C_{b,2}}{2}\end{bmatrix} \begin{bmatrix} i_b^{n-1}\\ v_{b,1}^{n-\frac{1}{2}}\\ v_{b,2}^{n-\frac{1}{2}}\end{bmatrix}\\ &= \frac{1}{2}\left(\mathbf{x}_b^n\right)^T\mathbf{P}_b\mathbf{x}_b^n. \end{aligned} \tag{23}$$

The quantity $E^n_b|_{(2,2)}$ is positive if the matrix $\mathbf{P}_b$ is positive definite. For $\mathbf{P}_b$ to be positive definite, all the upper left submatrices $\mathbf{P}_b^{(k)}$, where k denotes the size of upper left submatrix, should have positive determinants [9]. The determinant of the first upper left matrix should then satisfy

$$\left|\mathbf{P}_b^{(1)}\right| = L_b > 0. \tag{24}$$

So all branch inductances should be nonzero and positive. Similarly, it can be shown that the condition $|\mathbf{P}_b^{(2)}| > 0$ is true if

$$(\Delta t)^2 < 4L_b\frac{C_{b,1}}{2}. \tag{25}$$

Since $(\Delta t)^2$ is non-negative for any real Δt, from (25), it can concluded that $C_i > 0$, i.e., all capacitances to ground should be positive. Finally, it can be shown that the condition $|\mathbf{P}_b^{(3)}| > 0$ is true if

$$(\Delta t)^2 < 4L_b\frac{\frac{C_{b,1}}{2}\frac{C_{b,2}}{2}}{\frac{C_{b,1}}{2}+\frac{C_{b,2}}{2}}. \tag{26}$$

Making use of the fact that

$$\frac{C_iC_{i+1}}{C_i+C_{i+1}} \geq \frac{1}{2}\min(C_i, C_{i+1})$$

the condition in (26) is satisfied if

$$(\Delta t)^2 < 2L_b\min\left(\frac{C_{b,1}}{2}, \frac{C_{b,2}}{2}\right). \tag{27}$$

Since the condition in (27) is more strict than (25), the matrix $\mathbf{P}_b$ is positive definite if L's and C's are positive and (27) is satisfied. When this analysis is repeated for all b, the condition (27) becomes

$$\begin{aligned} (\Delta t)^2 &< 2\min_{b=1}^{N_b}\left(L_b\min\left(\frac{C_{b,1}}{2}, \frac{C_{b,2}}{2}\right)\right)\\ &< 2\min_{i=1}^{N_n}\left(\frac{C_i}{2}\min\left(L_{i+\frac{1}{2}}, L_{i-\frac{1}{2}}\right)\right) \end{aligned} \tag{28}$$

resulting in a condition for Δt as

$$\Delta t < \sqrt{2}\min_{i=1}^{N_n}\left(\sqrt{\frac{C_i}{2}\min\left(L_{i-\frac{1}{2}}, L_{i+\frac{1}{2}}\right)}\right). \tag{29}$$

When (29) is simplified, the well-known Courant time step for 1-D circuit is obtained:

$$\Delta t < \min_{i=1}^{N_n}\left(\sqrt{C_i\min\left(L_{i-\frac{1}{2}}, L_{i+\frac{1}{2}}\right)}\right). \tag{30}$$

If L's and C's are positive and Δt satisfies (29) or (30), E^n satisfies (10): If L's and C's are positive and Δt satisfies (29) or (30), $\mathbf{P}_b$ is positive definite, which makes $E^n_b|_{(2,2)}$ satisfy (10) [see (23)], which in turn makes E^n satisfy (10) (see (21)).

Since E^n is already shown to satisfy (11), from Section IV, the equilibrium state $\mathbf{u_e} = \mathbf{0}$ is stable. Now if $\mathbf{G} \neq \mathbf{0}$, which is satisfied if there is a node i such that $G_i \neq 0$, E^n also satisfies (12), making $\mathbf{u_e} = \mathbf{0}$ asymptotically stable. Moreover, it can be proven that if E^n in the form (20) is asymptotically stable, then E^n is also globally asymptotically stable. From Section IV, this proof is completed if E^n is shown to satisfy (13): the function E^n can be shown to be radially unbounded if $\mathbf{P}$ is positive definite [8, App. B], which is satisfied if E^n in the form (20) also satisfies (10). All radially unbounded functions satisfy (13)[8, App. B].

B. Condition on Δt When Arbitrary Number of Branches are Connected to a Node

Let N^i_b denote the number of branches connected to node i. The generic condition on Δt can be easily obtained by letting $p = N_b^{b,1}$ and $q = N_b^{b,2}$ in (22) and repeating the derivation

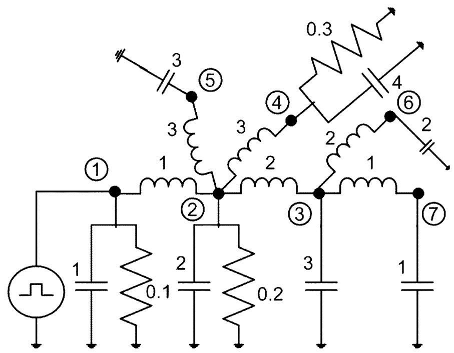

Fig. 2. Sample nonuniform G*LC* circuit. All resistors are in ohms, all capacitors in nanofarads, and all inductors in nanohenries.

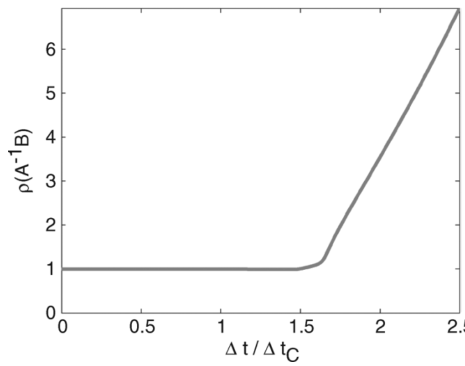

Fig. 3. Spectral radius of the matrix $\mathbf{A}^{-1}\mathbf{B}$ is shown as a function of the ratio $\Delta t/\Delta t_C$.

from (21)through (29). For a generic case, the condition on Δt in (29) can be shown to be

$$\Delta t < \sqrt{2} \min_{i=1}^{N_n} \left(\sqrt{\frac{C_i}{N_b^i} \min_{p=1}^{N_b^i} \left(L_{\langle i,p \rangle} \right)} \right) \tag{31}$$

where $L_{\langle i,p \rangle}$ denotes the value of pth inductor connected to node i. As can be observed, the derivation described thus far does not require the circuit to be uniform or infinitely long. Also, the (equivalent) circuits can be discontinuous, i.e., have irregularities in connections to neighboring nodes. Such discontinuities are observed in irregular on-chip power grids or in package power/ground planes with a hole.

VI. Limitations

The Lyapunov function (18) proposed for G*LC* circuits in this paper has a dual drawback to the Lyapunov function proposed for *RLC* circuits in [4]: The function (18) does not remain a Lyapunov when nonzero series R's are added to the inductors in the G*LC* circuits. As a result, analytical stability condition of LIM for nonuniform RLGC circuits is still an open problem.

The proposed analytical stability condition is valid only for linear circuit elements as shown in Fig. 1. In case of a nonlinear circuit element, such as a diode to ground, the update expression (14) changes; these changes can become time dependent. Consequently, this analysis (which gives a time-independent stability condition) remains not proven when nonlinear elements are present.

VII. Numerical Validation

In this section, the sufficient condition for stability provided by the choice of Δt according to (31) is numerically verified. Let Δt_C denote the quantity on the right-hand side of (31). Since the LIM iteration in (7) is going to converge when $\rho(\mathbf{A}^{-1}\mathbf{B}) \leq 1$, then the stability is verified if the condition $\rho(\mathbf{A}^{-1}\mathbf{B}) \leq 1$ is verified for $\Delta t < \Delta t_C$.

Consider a sample circuit shown in Fig. 2. The spectral radius, $\rho(\mathbf{A}^{-1}\mathbf{B})$, is computed for different values of Δt and is plotted in Fig. 3. For small Δt, $\mathbf{A}^{-1}\mathbf{B}$ becomes close to an identify matrix [see (8) and (9)], making its spectral radius close to unity. This unity spectral radius can be observed in Fig. 3 as well. Also, it can be observed that $\rho(\mathbf{A}^{-1}\mathbf{B}) \leq 1$ when $\Delta t < \Delta t_C$. Also, it can be observed that as Δt increases, $\rho(\mathbf{A}^{-1}\mathbf{B}) > 1$, demonstrating the conditional nature of the LIM's stability. Also, it can be noticed that when Δt is slightly greater than Δt_C, the $\rho(\mathbf{A}^{-1}\mathbf{B}) \leq 1$ still. This demonstrates the sufficient condition nature of (31).

VIII. Conclusion

The LIM is a transient simulation technique for circuits and is based on a finite-difference formulation, like the well-known FDTD method for solving Maxwell's equations. The LIM, like the FDTD method, is only conditionally stable, resulting in an upper bound for the time step of the transient simulation. This bound on the time step is a function of the circuit topology and circuit element values. It is critical to know this bound analytically for a given circuit. Stability conditions of the LIM have been proven only for nonuniform *RLC* circuits. In this paper, analytical stability condition of the LIM for nonuniform G*LC* circuit is derived in this paper, resulting in an upper bound for the time step. This upper bound for time step is verified numerically.

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